

PRIMES AND THEIR DISTRIBUTION

Definition: Let p be an integer greater than 1, then p is said to be a prime number or simply a prime if it has 1 and itself as its only positive divisors.

Observations:

- (i) The distribution or position of the prime numbers in the real number line do not have distinguishable pattern unlike the set of even or odd integers that follow an alternating pattern in the real number line.
- (ii) 2 is the least of all prime numbers and is the only even prime number, the rest are all odd prime numbers.
- (iii) If p is not prime, then p is said to be a composite number (numbers that have at least one more positive divisor than that of a prime number.)

INVESTIGATIONS ON PRIME NUMBERS:

1) If a is composite and $q \neq 1$ is a least positive divisor of a , then $q \leq \sqrt{a}$.

Example: Is 221 prime or composite?

Solution:

Since $q \leq \sqrt{221}$ if 221 is composite, therefore:

$$14^2 = 196$$

$$15^2 = 225$$

Thus $q = 2, 3, \dots, 14$.

By **Division Algorithm**, we have:

$$(i) \quad 221 \div 2$$

$$221 = 2(110) + 1$$

Since the remainder is $1 \neq 0$, therefore $2 \nmid 221$.

$$(ii) \quad 221 \div 3$$

$$221 = 3(73) + 2$$

Since the remainder is $2 \neq 0$, therefore $3 \nmid 221$.

$$\begin{aligned} \text{(iii)} \quad & 221 \div 4 \\ & 221 = 4(55) + 1 \end{aligned}$$

Since the remainder $1 \neq 0$, therefore $4 \nmid 221$.

$$\begin{aligned} \text{(iv)} \quad & 221 \div 5 \\ & 221 = 5(44) + 1 \end{aligned}$$

Since the remainder is $1 \neq 0$, therefore $5 \nmid 221$.

$$\begin{aligned} \text{(v)} \quad & 221 \div 6 \\ & 221 = 6(36) + 5 \end{aligned}$$

Since the remainder is $5 \neq 0$, therefore $6 \nmid 221$.

$$\begin{aligned} \text{(vi)} \quad & 221 \div 7 \\ & 221 = 7(31) + 4 \end{aligned}$$

Since the remainder is $4 \neq 0$, therefore $7 \nmid 221$.

$$\begin{aligned} \text{(vii)} \quad & 221 \div 8 \\ & 221 = 8(27) + 5 \end{aligned}$$

Since the remainder is $5 \neq 0$, therefore $8 \nmid 221$.

$$\begin{aligned} \text{(viii)} \quad & 221 \div 9 \\ & 221 = 9(24) + 5 \end{aligned}$$

Since the remainder is $5 \neq 0$, therefore $9 \nmid 221$.

$$\begin{aligned} \text{(ix)} \quad & 221 \div 10 \\ & 221 = 10(22) + 1 \end{aligned}$$

Since the remainder is $1 \neq 0$, therefore $10 \nmid 221$.

$$\begin{aligned} \text{(x)} \quad & 221 \div 11 \\ & 221 = 11(20) + 1 \end{aligned}$$

Since the remainder is $1 \neq 0$ therefore $11 \nmid 221$.

$$\begin{aligned} \text{(xi)} \quad & 221 \div 12 \\ & 221 = 12(18) + 5 \end{aligned}$$

Since the remainder is $5 \neq 0$, therefore $12 \nmid 221$.

$$\begin{aligned} \text{(xii)} \quad & 221 \div 13 \\ & 221 = 13(17) + 0 \end{aligned}$$

Since the remainder is 0, therefore $13 \mid 221$.

Answer: Since $221 \div 13 = 17$ and $q = 13 \leq \sqrt{221}$, therefore 221 is composite.

1) If m is composite, then $n_m = 111 \dots 1$ (m times) is also composite.

Example: $n_6 = 111 111$ is composite since $m = 6$ is composite.

- To verify that $n_6 = 111 111$ is indeed composite, we use the first investigation of primes.

Solution:

$$q \leq \sqrt{111 111} \approx 333.33$$

$$\rightarrow q = 2, 3, \dots 333.$$

By **Division Algorithm**, we have:

$$(i) \quad \begin{aligned} 111 111 &\div 2 \\ 111 111 &= (55555) + 1 \end{aligned}$$

Since the remainder is $1 \neq 0$, therefore $2 \nmid 111 111$

$$(ii) \quad \begin{aligned} 111 111 &\div 3 \\ 111 111 &= 3(37037) + 0 \end{aligned}$$

Since the remainder is 0, therefore $3 \mid 111 111$.

Answer: Since $111 111 \div 3$ and $q = 3 \leq \sqrt{111 111}$, therefore $n_6 = 111 111$ is composite.

Remark: Investigation / Statement 2 does not imply that if m is prime n_m is prime.

An example is $n_3 = 111 = 3(37) + 0$. $m = 3$ is prime but $n_3 = 111$ is composite.

Example: $n_5 = 11111$, $m = 5$ is prime but 11111 is composite

Since 11111 has divisors 1, 41, 271, 11111 For every integer $n > 1$,

there are n consecutive composite numbers.

3) For every integer $n > 1$, there are n consecutive composite numbers.

Example: When $n = 2$, we have 2 consecutive composite numbers 8, 9 or 14, 15

- $n = 3$: 14, 15, 16 or 8, 9, 10 or 20, 21, 22

- $n = 4$: 24, 25, 26, 27
- $n = 5$: 24, 25, 26, 27, 28

Consider the n consecutive integers.

$(n + 1)! + 2, (n + 1)! + 3, \dots (n + 1)! + i$ when $2 \leq i \leq n + 1$

we know that $i \mid (n + 1)! + i$ by **Corollary**: If $a \mid b$ and $a \mid c$, then $a \mid b + c$.

Example: $(2 + 1)! + 2 = 6 + 2 = 8 \rightarrow i = 2$, therefore

$$i \mid (2 + 1)! + i$$

$$2 \mid (2 + 1)! + 2$$

$$2 \mid 6 + 2$$

$$2 \mid 8$$

4) Euclid's Theorem: The number of primes is infinite.

Trivia: The Greatest Prime Number as of February 2023 is $2^{82,589,933} - 1$

Recall Euclid's Lemma: If $m \mid (a.b)$ and $(a, m) = 1$, then $m \mid b$.

Theorem: If p is prime and $p \mid a.b$, then $p \mid a$ or $p \mid b$.

Proof: Given: p is prime and $p \mid a.b$.

Suppose p is prime and so its only positive divisors are 1 and p . If $(p, a) = p$, then $p \mid a$.

Suppose p is prime but $p \nmid a$. Then $(p, a) = 1$. Thus, by definition of Euclid's Lemma, $p \mid b$.

Corollary a): If p is prime and $p \mid a_1 a_2 \dots a_n$, then $p \mid a_k$ for some $k, 1 \leq k \leq n$.

Example: $7 \mid 336$

$$\rightarrow 7 \mid 6(56) \rightarrow 7 \mid 56$$

$$\rightarrow 7 \mid 6(7)(8) \rightarrow 7 \mid 7$$

Corollary b): If p, q_1, q_2, \dots, q_n are all primes and $p \mid q_1 q_2 \dots q_n$, then $p = q_k$ for some $k, 1 \leq k \leq n$.

Example: $3|210$

$$3|2(105) \rightarrow 210 = 2(105)$$

$$3|2(3)(35) \rightarrow 105 = (3)(35)$$

$$3|2(3)(5)(7) \rightarrow 35 = (5)(7)$$

Since 2,3,5,7 are all primes and $3|210$ or $3|2(3)(5)(7)$.

Therefore $3|3$ or $3 = 3$.